

A note on certain convolution operators

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Abstract

In the note we consider convolution operators acting on the L_1 space of functions defined on the unit circle equipped with the Lebesgue measure. Kernels are taken to be densities of real random variables with characteristic functions belonging to some Lebesgue space L_p . We prove that the identity minus such an operator is nicely invertible on the subspace of functions with mean zero.

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1 Introduction

This paper is devoted to a study of a certain class of convolution operators $A : L_1(\mathbb{T}) \rightarrow L_1(\mathbb{T})$, where \mathbb{T} is the one dimensional torus. The question about properties of this kind of operators arose during the work on the following question posted by Gideon Schechtman (personal communication).

Question. Given $\varepsilon > 0$, is it true that there exists a natural number $k = k(\varepsilon)$ such that for any bounded linear operator $T : L_1[0, 1] \rightarrow L_1[0, 1]$ with $\|T\|_{L_1 \rightarrow L_1} \leq 1$ which has the property

$$\forall f \in L_1[0, 1] \quad (|\text{supp } f| \leq 1/2 \implies \|Tf\| \geq \varepsilon \|f\|)$$

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there exist $\delta > 0$ and functions $g_1, \dots, g_k \in L_\infty[0, 1]$ such that

$$\|Tf\| \geq \delta\|f\| \quad \text{for any } f \in L_1[0, 1] \text{ satisfying } \int_0^1 f g_j = 0, j = 1, \dots, k?$$

Here and throughout, we use the notation $\|\cdot\| = \|\cdot\|_{L_1(\mathbb{T})}$. It is worth to mention that the question in an equivalent form was asked by Bill Johnson in relation with a question on Mathoverflow [MO].

Hoping to give a negative answer to this question we were considering operators of the form $T = I - B_Y$, where

$$(B_Y f)(x) = \mathbb{E}f(x \oplus Y), \quad \text{for } x \in \mathbb{T}. \quad (1)$$

Here Y is a real random variable, $\mathbb{T} = [0, 1)$ and $x \oplus y = (x + y) \bmod 1$. Our first idea was to take an operator

$$(U_t f)(x) = \frac{1}{2t} \int_{-t}^{+t} f(x \oplus s) \, ds, \quad t \in (0, 1). \quad (2)$$

Clearly, we have $U_t = B_{tY}$, where Y is uniformly distributed on the interval $[-1, 1]$. It turns out that the operators of the form (2) will not provide the negative answer to Schechtman's question as we were able to prove the following theorem.

Theorem 1. *Consider $t \in (0, 1)$ and $f \in L_1(\mathbb{T})$ with $\int_{\mathbb{T}} f = 0$. Then we have*

$$\|f - U_t f\| \geq ct^2 \|f\|,$$

where $c > 0$ is a universal constant.

Indeed, the assertion says that the operator $T = I - U_t$ is nicely invertible on the subspace of functions such that $\int f \cdot 1 = 0$. That is why T cannot serve as the negative answer to Schechtman's question.

Remark. Set $f(x) = \cos(2\pi x)$. Then $\|f - U_t f\| = \frac{2}{\pi} \left(1 - \frac{1}{2\pi t} \sin(2\pi t)\right) \approx t^2$, for small t . Therefore the inequality in Theorem 1 is sharp in a sense.

In this short note we give a proof of a generalization of this theorem. Namely, the result reads

Theorem 2. *Given $t \in (0, 1)$ and a real random variable Y with characteristic function belonging to $L_p(\mathbb{R})$ for some $p \geq 1$, consider the operator $A_t : L_1(\mathbb{T}) \rightarrow L_1(\mathbb{T})$ given by*

$$(A_t f)(x) = \mathbb{E}f(x \oplus tY).$$

Then for all $f : \mathbb{T} \rightarrow \mathbb{T}$ with $\int_{\mathbb{T}} f = 0$ we have

$$\|f - A_t f\| \geq ct^2 \|f\|,$$

where a positive constant c depends only on the distribution of the random variable Y .

Remark. For instance, if Y has bounded density, then its characteristic function is in L_2 .

The next section is devoted to the proof of Theorem 2.

2 Proof of Theorem 2

We begin with two lemmas.

Lemma 1. *Suppose Y is a real random variable with characteristic function belonging to $L_p(\mathbb{R})$ for some $p \geq 1$. Let Y_1, Y_2, \dots be independent copies of Y . Then there exists a positive integer $N = N(Y)$ and a number $c = c(Y) > 0$ such that for all $C \geq 1$ and $n \geq N$ the density of*

$$X_n^{(C)} = \left(C \cdot \frac{Y_1 + \dots + Y_n}{\sqrt{n}} \right) \mod 1$$

is bounded below by c .

Proof. By a certain version of the Local Central Limit Theorem, e.g. Theorem 19.1 in [BR], p. 189, we know that the density q_n of $(Y_1 + \dots + Y_n - n\mathbb{E}Y)/\sqrt{n}$ exists for sufficiently large n , and satisfies

$$\sup_{x \in \mathbb{R}} \left| q_n(x) - \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \right| \xrightarrow{n \rightarrow \infty} 0, \quad (3)$$

where $\sigma^2 = \text{Var}(Y)$. Observe that the density $g_n^{(C)}$ of $X_n^{(C)}$ equals

$$g_n^{(C)}(x) = \sum_{k \in \mathbb{Z}} \frac{1}{C} q_n \left(\frac{1}{C} (x + k + \sqrt{n}\mathbb{E}Y) \right), \quad x \in [0, 1].$$

Using (3), for $\delta = \frac{e^{-2/\sigma^2}}{\sqrt{2\pi}\sigma}$ we can find $N = N(Y)$ such that

$$q_n(x) > \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} - \delta/8, \quad x \in \mathbb{R}, \quad n \geq N.$$

Therefore, to be close to the maximum of the Gaussian density we sum over only those ks for which $x + k + \sqrt{n}\mathbb{E}Y \in (-2C, 2C)$ for all $x \in [0, 1]$. Since there are at least C and at most $4C$ such ks , we get that

$$g_n^{(C)}(x) > \frac{1}{C} \frac{1}{\sqrt{2\pi}\sigma} e^{-2/\sigma^2} \cdot C - \frac{1}{C} \frac{\delta}{8} \cdot 4C = \frac{1}{2\sqrt{2\pi}\sigma} e^{-2/\sigma^2}.$$

□

Lemma 2. *Suppose Z is a \mathbb{T} -valued random variable with a density h bounded below by $c \in (0, 1)$ and B_Z is defined by (1). Then $\|B_Z f\| \leq (1 - c)\|f\|$ for all $f \in L_1(\mathbb{T})$ satisfying $\int_{\mathbb{T}} f = 0$.*

Proof. Take $f \in L_1(\mathbb{T})$ with zero mean. We then have

$$\begin{aligned} \|B_Y f\| &= \int_0^1 \left| \int_0^1 h(s) f(x \oplus s) \, ds \right| dx = \int_0^1 \left| \int_0^1 (h(s) - c) f(x \oplus s) \, ds \right| dx \\ &\leq \int_0^1 \int_0^1 (h(s) - c) |f(x \oplus s)| \, ds \, dx \\ &= \|f\| \int_0^1 (h(s) - c) \, ds = (1 - c)\|f\|. \end{aligned}$$

□

Now we are ready to give the proof of Theorem 2.

Proof of Theorem 2. Let Y_1, Y_2, \dots be independent copies of Y . Observe that

$$\begin{aligned} (A_t^n f)(x) &= \mathbb{E} f(x \oplus tY_1 \oplus \dots \oplus tY_n) \\ &= \mathbb{E} f\left(x \oplus \left(t\sqrt{n} \left(\frac{Y_1 + \dots + Y_n}{\sqrt{n}}\right) \bmod 1\right)\right). \end{aligned}$$

Take $n(t) = \lceil 1/t^2 \rceil N(Y)$, where $N(Y)$ is the number given by Lemma 1. Therefore

$$(A_t^{n(t)} f)(x) = \mathbb{E} f\left(x \oplus X_{n(t)}^{(C)}\right),$$

where $C = t\sqrt{n(t)} = t\sqrt{\lceil 1/t^2 \rceil N(Y)} \geq \sqrt{N(Y)} \geq 1$. Thus $X_{n(t)}^{(C)}$ has a density bounded below by some constant $c(Y) \in (0, 1)$. From Lemma 2 we have

$$\|A_t^{n(t)} f\| \leq (1 - c(Y)) \|f\|$$

for all f satisfying $\int_{\mathbb{T}} f = 0$.

The operator A_f is a contraction, namely $\|A_t f\| \leq \|f\|$ for all $f \in L_1(\mathbb{T})$. Using this observation and the triangle inequality we obtain

$$\begin{aligned} \|f - A_t f\| &\geq \frac{1}{n} (\|f - A_t f\| + \|A_t f - A_t^2 f\| + \dots + \|A_t^{n-1} f - A_t^n f\|) \\ &\geq \frac{1}{n} \|f - A_t^n f\|. \end{aligned}$$

Taking $n = n(t)$ we arrive at

$$\frac{1}{n(t)} \|f - A_t^{n(t)} f\| \geq \frac{1}{t^{-2} + 1} \cdot \frac{1}{N(Y)} (\|f\| - \|A_t^{n(t)} f\|) \geq \frac{c(Y)}{2N(Y)} t^2 \|f\|.$$

It suffices to take $c = c(Y)/2N(Y)$. □

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